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Volume and Surface Integrals used in Physics

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VOLUME AND SURFACE INTEGRALS USED IN PHYSICS

by

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CONTENTS.

gravitation property, then density is a term having a precise meaning; but if the distribution of the gravitation property through the space occupied by the body has not this mathematical continuity, we cannot attach any meaning to the volume integrals till we have first invented a suitable new meaning for the term 'density'; and the inevitable vagueness that will arise in the new definition will preserve in the final results that slight lack of precision present in the terms of statement of the gravitation law, which might at first sight appear to have dropped out of results represented by such precise mathematical expressions as volume integrals.

It is practically certain that no substance can be subdivided without limit into small portions each of which possesses the gravitation property. There must be a stage of subdivision beyond which the component portions cease to have the properties of larger portions of the substance, and we may speak of the smallest portion of a substance that has the gravitation property as a 'particle' for purposes of the present discussion. What the order of magnitude of a particle may be it is difficult to guess, but the kind of generalisation from large bodies to small bodies which led to the conception of an element of mass suggests the possibility that the process of subdivision without loss of the gravitation property might be continued till we arrive at the molecule of Chemistry or Gas Theory. There is no experimental evidence to prohibit, and possibly some to justify our carrying the generalisation so far (provided we set some limit to the smallness of the distance at which the attraction of two particles is supposed to obey the law of the inverse square), and the great simplicity of the law thus obtained makes it an interesting one to study.

If a body has not got mathematical continuity in the distribution of the gravitation property throughout its volume, and so is to be regarded as made up of particles, we may speak of it as having 'discontinuous' structure. And if it be supposed that the particles may be as small as molecules, we must form a mental picture of the structure in which there appears no trace of material continuity, the substance being represented by discrete molecules or systems occupying spaces with somewhat indefinite boundaries, separated by more or less empty regions which may be called intermolecular space; (if the molecules move it may be supposed that their motions do not affect the properties under consideration). The problem of finding the potential or attraction of such a body at any point, if formulated on a greatly magnified and coarser scale, would in some respects resemble

the problem of evaluating the potential or attraction of a mass of sand or other granular matter. We want to see how volume integrals present themselves as approximate solutions of such problems.

It is unnecessary to dwell here upon the familiar definitions of the intensity of force at a point, or the potential at a point, due to a gravitating body. But, for future reference, we may emphasize the fact that, among the mathematical properties of the potential at a point outside the body, that which may be taken as the fundamental physical definition is the fact that its space-gradient at any point is vectorially equal to the intensity of force there. If a point is so situated that a physical definition of intensity of force there is impossible, this physical definition of potential breaks down, and we are at liberty to substitute some convenient purely mathematical definition for which it may be possible afterwards to find a physical interpretation.

The potential of a body of discontinuous structure at an external point P is the sum of the potentials at P due to the particles that compose the body, i.e. Σmr^{-1}, where m is the mass of a particle and r its distance from P. Here, in accordance with what has been said above, r is not precisely defined, and a corresponding lack of precision must be present in Σmr^{-1}. By assuming P to be not too close to any particle of the body we can ensure that each r shall always be great compared with the linear dimensions of the corresponding particle.

When we endeavour to compare the values of this expression for the potential at different points, we recognise that the sum of a finite but extremely great number of extremely small terms is a most troublesome function to work with, and so there naturally suggests itself the device of getting a probably very approximate equivalent function by replacing the sum by the limit to which it would tend if the number of terms could be increased indefinitely while each separate term decreased correspondingly; this process would give us the potential in the form of the definite integral $\int \rho r^{-1} d\tau$, where $d\tau$ is an element of volume and $\rho d\tau$ the corresponding mass.

But, as has been suggested already, the transition from a sum of terms to a definite integral would imply the possibility of *endless* subdivision of the material mass into elements each possessing the gravitation property, whereas it is practically certain that matter cannot be so endlessly subdivided. In fact the use of the definite integral form implies a regarding of matter as continuously extended through the space which it effectively occupies, and attributes to the

density ρ at any point the value obtained by passing to a mathematical limit in the usual fashion, that is to say the limit of the quotient of mass by volume for a region surrounding the point as the dimensions of the region tend to zero. The molecular view, however, requires us to cease subdividing matter beyond a certain stage, and so prevents our ever arriving at the kind of limit which is known as an integral.

3. Nevertheless the potential at the point P of an assemblage of discrete particles in a finite region may be equal in value to a volume integral taken throughout the region if the integral be supposed to refer to a hypothetical continuous medium occupying the same region and having a suitably chosen density at each point. It is only necessary to choose the law of density properly, and to this end there suggests itself the device of taking for each point A some sort of average density, based on a consideration of all the masses within a very small but finite region surrounding A. The dimensions of this small region might be settled by convention, but we need only consider the order of magnitude of these dimensions.

The kind of smallness that we want in this connexion is what we may call *physical smallness*, as distinguished from mathematical smallness to which there is no limit. Physical smallness implies smallness which appears extreme to the human senses, but it must not be a smallness so extreme as to necessitate passage from molar physics to molecular physics; it must leave us at liberty, for example, to attribute to matter occupying a physically small space the properties of matter in bulk if these should be different from the properties of isolated molecules. In fact a physically small region, though extremely small, must still be large enough to contain a very great number of molecules. It is estimated that a gas, at normal temperature and pressure, has about 4×10^{19} molecules per cubic centimetre; thus a cube whose edge is 6×10^{-5} centimetres (roughly the wave length of sodium light) contains more than 8,000,000 molecules; if we regard a million as a large number, the wave length of sodium light is (for other than optical purposes) physically small, and it is known that very much greater lengths than this appear to our senses extremely small. We are therefore in a position to speak of lengths which, though extremely small, are very great compared with other physically small lengths.

Now it is not suggested that the gravitation property is a molar property of matter, not possessed by a single molecule, for we have adopted just the opposite hypothesis; and so it might be thought justifiable to make the region round A, used for calculating ρ,

smaller than merely physically small. This point will be referred to again, but at present it suffices to remark that the ρ generally used in potential theory is a continuous function whose value is not subject to very rapid fluctuations as A moves from one position to another. To get such continuity and smoothness in the suggested average, and to avoid the difficulty that would arise if the number of molecules intersected by the boundary of the region so as to be neither obviously inside nor obviously outside were not very small in comparison with the total number inside, we must take account of a large number of molecules at a time, and so we have to assume that the region surrounding the point A and used in getting a value for ρ is only physically small.

As regards the system of averaging, it is clear that in the case under discussion we get the best agreement between the potential of the actual and that of the hypothetical system if we give to each molecule an importance proportional to the product of its mass and the reciprocal of its distance from P; but it would be unfortunate to be obliged to average in this fashion, as we should thereby get a value of ρ at A which would not be independent of the position of P. And we should get quite a different law of density if we were dealing with some other integral, say an attraction integral, instead of that representing potential.

So long, however, as P is at a distance from A great compared with the linear dimensions of the physically small region used for the purpose of averaging, the values of r^{-1} for the molecules in this region are very nearly equal, and so there is very little error if in taking the average we give to each molecule an importance simply proportional to its mass. We thus get for the density ρ of the hypothetical continuous medium the quotient of mass by volume for the physically small region considered. This value of ρ has the advantage that it is independent of the position of P, and of the particular physical quantity whose integral expression is being investigated. But its great advantage, and the real reason why we adopt it, is that it is that density of a substance which we actually arrive at by practical methods of measurement; for ordinary laboratory measurings and weighings are applied to portions of a substance which are far from the limits of physical smallness, and so give us, not the sizes and masses of individual molecules, but only the total mass and the total space effectively occupied.

4. So far we have considered only the case in which the point P at which the potential (or other function of position) is to be estimated

is at a distance from the nearest portion of the gravitating mass great compared with the linear dimensions of what we have called a physically small region. Such a distance, which, as we have seen, may be extremely small compared with the smallest distance we can measure directly, would seem to mark the limit of nearness of P to the gravitating body if the integral taken for the hypothetical continuous medium is to serve as equivalent to the true potential. But further consideration may enable us to push this limit still closer to the body. For the inaccuracies whose importance is magnified by decreasing distance do not, for a given position of P, occur in the case of each molecule of the body; they arise only in connexion with the molecules that are near P. Now such molecules, though perhaps absolutely numerous, are generally few in comparison with the remaining molecules of the body, and it is possible that their numerical inferiority may prevail over the advantage of their position in such a way as to render the total inaccuracy corresponding to them a negligibly small fraction of the whole potential.

Instead of the function r^{-1} which occurs in expressions for the potential, let us consider some other function f of position relative to P, which tends, as r becomes smaller, to become infinite of the same order as $r^{-\mu}$, so that, for small values of r, f is of the form $kr^{-\mu}$ where k is a finite function of relative angular position. Taking, in the first instance, a single physically small element of matter, say of volume ϵ^3, it is clear that the difference between the sum Σmf and the integral $\int \rho f d\tau$ though the element will in general be of the same order of magnitude as either quantity separately so long as r for all points of the element is of the same order of magnitude as ϵ, but that the difference will diminish to a quantity smaller in a ratio comparable with ϵr^{-1} when r becomes great compared with ϵ. Hence, for purposes of estimating order of magnitude, it is fair to represent the difference between Σmf and $\int \rho f d\tau$ by the expression $\int A\epsilon r^{-1}\rho f d\tau$ where A is a finite number.

To include all elements of the body near to P, we suppose the least value of r for a molecule near to P to be η, and take the integral representing inaccuracy through all space between the concentric spheres $r=\eta$ and $r=a$, where a is large compared with ϵ. If ρ' is the greatest value of ρ in this space, the order of magnitude of the inaccuracy is the same as or less than that of

$$\rho' k' A \int_{\eta}^{a} \epsilon r^{-1} r^{-\mu} 4\pi r^2 dr,$$

where k' is a finite constant replacing k, and $4\pi r^2 dr$, the volume between the spheres of radii r and $r + dr$, takes the place of $d\tau$. This is equal to

$$4\pi\rho' k' A \epsilon \{a^{2-\mu} - \eta^{2-\mu}\}/(2-\mu),$$

of which, when η is of the same order as ϵ, the first term is small of order ϵ and therefore negligible always, while the second term is small of the same or higher order provided $\mu < 2$; the second term would be small, but not of so high an order of smallness, if μ were between 2 and 3. The case of $\mu = 2$ would turn on the order of magnitude of $\epsilon \log \eta$ or $\epsilon \log \epsilon$, which is very small though not as small as ϵ. Sometimes the special form of the function k, taken in connexion with probable symmetry in the average distribution of molecules round P, increases still further the order of smallness.

It follows, therefore, that if $\mu < 2$ the inaccuracy is certainly as negligible as ϵb^{-1}, and that if $\mu < 2\frac{1}{2}$ the inaccuracy is certainly as negligible as $\sqrt{\epsilon b^{-1}}$, where b is some length which is physically not small, e.g. a centimetre. The case of $\mu = 1$ corresponds to potential; for attraction components $\mu = 2$.

5. Thus it appears that the representation of potentials and attractions by means of integrals extended through the hypothetical continuous medium which replaces the actual gravitating body is valid without sensible error not only for points well outside the body, but also for points whose distance from the nearest portion of the body is small of the order of the physically small length ϵ. This includes the case when P is so close to the apparent outer surface of the body as to be sensibly just not in contact with it, and also the case when P is in a small but not imperceptibly small cavity cut in the body, that is a cavity of such a size that the piece excavated would have the properties of matter in bulk rather than the properties of a few molecules.

As might be expected, any attempt to justify the use of the same integral expressions for the potential and attractions at a point P which is at a distance from the nearest molecule of a higher order of smallness than ϵ, results in failure. For now a simple molecule contributes to the potential (for example) a term mr^{-1} which, in spite of the smallness of m, may become very great as r diminishes; how small r may become we cannot say, its least possible value must depend on the extent to which the 'impenetrability' of matter is true of isolated molecules, for, since potential is only physically interpretable as the negative potential energy per unit mass of a particle (at least one molecule) at P, the least value of r is the least possible

distance between the centres of two molecules. While not knowing this least value, we cannot but admit the possibility that a few terms of the type mr^{-1} might easily become so important as to make the potential quite different from the value of $\int \rho r^{-1} d\tau$ to which, as we shall see later, the parts of the hypothetical continuous distribution near P contribute only a negligible amount. But there is in any case, from the point of view of physics, no motive for pursuing the enquiry to such small values of r, for there are reasons for supposing that the Newtonian law of attraction does not hold good at such distances. In proving that the ordinary integral representations of potential and attractions are valid for distances of P from the attracting body which are indefinitely small from the point of view of molar physics, we have done all that would be required to justify their ordinary use in the theory of gravitation.

6. One point requires emphasis. By the attraction at a point P inside a body we mean the attraction (per unit mass) on a molecule or particle at P, situated in a cavity of dimensions which are only physically small. Hence if we describe a closed geometrical surface S, however small, in a body, we cannot calculate the force exerted on the part of the body inside S by the rest of the body by use of the attraction integrals. This can only be done when there is a gap, not smaller than physically small, between the attracting and the attracted matter, such a gap as might be made by cutting the body along the surface S and keeping the fissure open so that the opposite sides of it are nowhere in contact. In the absence of such a physical separation, account must be taken of unknown forces between molecules that are very near together. When the volume enclosed by S is not small beyond the physical limit of smallness, such molecules will all lie relatively near the surface S, and the forces between them will appear as surface forces between the geometrically separated portions of the body. In fluids the extra force is the fluid pressure, in solids it is less simple.

In the case of an absolutely continuous body there is nothing corresponding to the limit of physical smallness, and if the Newtonian law were supposed to hold for all distances however small there would be no surface forces of the kind described. The assumption of surface forces in the ideal case of continuity is really a tacit assumption that the Newtonian law breaks down ultimately as r diminishes.

7. We might, of course, devise other definite integrals than those above considered, in the hope of representing the same physical

quantities with possibly greater accuracy. For example ρ might be obtained by averaging through a region smaller than physically small, so small that the number of molecules in it might be sometimes two or one or even zero; in this case fractions of molecules would become important, and the question would arise how a molecule ought to be regarded when it is neither altogether inside nor altogether outside the region. Again we might reduce the region of averaging to the limit of mathematical smallness and so get a ρ which is absolutely zero in intermolecular space, and presumably finite and continuous in the spaces occupied by the various molecules. Against such integrals it is to be urged firstly that one important factor of the function to be integrated, namely ρ, is inaccessible to experimental measurement, secondly that even if ρ were known the integrals would probably be more difficult to evaluate than the sum Σmr^{-1}, and thirdly that the greater accuracy which they seem to possess would be entirely vitiated by the probable failure of the Newtonian law for short distances. Moreover a method which involves integrating through the volumes of individual molecules, if it has any physical significance at all, implies the view that a molecule is of the nature of a small continuous mass whose smallest parts have the same kind of properties as the whole; this view is directly contrary to modern views of the constitution of matter, and the mathematical method corresponding to it, so far from being the best possible representation of the facts, must share all the defects of the method of summation for the various molecules.

II. Potentials and Attractions of accurately continuous bodies.

8. The potential and the attraction components of a finite body of accurately continuous substance, at an external point P, are represented by volume integrals which, for ordinary laws of density, give rise to no mathematical difficulties. The subjects of integration are finite at all points of the region of integration, and the integrals themselves are finite and differentiable with respect to the coordinates of P by the method known as 'differentiation under the sign of integration.' Thus the potential integral, defined as $\int \rho r^{-1} d\tau$, justifies its right to the name 'potential' by possessing the property that its differential coefficients with respect to the coordinates (ξ, η, ζ) of P are the attraction integrals of the type $\int \rho (x - \xi) r^{-3} d\tau$.

But it is quite another thing when we come to consider the potential and attractions at a point inside the gravitating body. For now, for example, if we define the potential as $\int \rho r^{-1} d\tau$ taken throughout the whole body, the subject of integration ρr^{-1} becomes infinite at the point P, a point in the volume of integration, and it becomes a question whether the integral symbol represents a finite quantity at all, and, if so, whether it is differentiable and what are its differential coefficients.

These troublesome questions might be avoided by introducing, as in the investigation for a body of molecular structure, a cavity within which P must be situated. And, indeed, this still seems to be demanded by the physical interpretation, since potential and attraction are physically defined as work function and force per unit mass for a hypothetical small mass or particle at the point P; such particle cannot be supposed to occupy space already occupied by other matter, and hence must be situated in a cavity made for it. But whereas, in the case of molecular structure, there was suggested from physical considerations a limit to the order of smallness of the cavity contemplated, corresponding in fact to the order of smallness of the necessarily present inaccuracy in the mathematical representation adopted, no such limit suggests itself in the case of continuous bodies. The retention of a cavity, of any definite though arbitrarily chosen order of smallness, is not demanded when there is no limit to the possible smallness of a portion of matter, and would moreover involve a want of precision or at least a restriction on the meaning of the mathematical symbols employed which would considerably discount their utility. Whereas it is only for the sake of mathematical precision that the hypothetical continuous bodies are generally made the subject of study in preference to the actual molecular bodies of which they are approximate representations.

9. We obtain the definiteness we desire, and, as will be seen, conform at the same time to the conventions and definitions of Integral Calculus, by framing new definitions of the potential and the attraction components at a point P (ξ, η, ζ), inside a continuous body. We first suppose the point P to be in a cavity, we then make the cavity smaller and smaller, and define the *limits* (if such exist) to which the potential and the attraction components at P tend with the vanishing of the cavity as the potential and the attraction components respectively at P when no cavity exists. It must be recognised that this passage to the limit entirely destroys the physical meaning which the quantities considered possess at any stage short of the limit, but on the other

hand it gives us extremely convenient standard approximations to these quantities in cases of physical interest; the very definition of the term limit implies that the approximation can be made as close as we please by taking the cavity sufficiently small.

It is also to be noticed that a relation such as $X = \frac{\partial V}{\partial \xi}$ (where X is a force component and V the potential), which holds inside a cavity of finite size however small, might not persist after passage to the limit. That is to say, though of necessity

$$\text{Lim } X = \text{Lim } \frac{\partial V}{\partial \xi},$$

it is not equally inevitable that

$$\text{Lim } X = \frac{\partial}{\partial \xi} \text{Lim } V.$$

In fact if, as is customary, we drop the phrase 'limit' from our notation, though keeping the idea in mind, we have to face the fact that the formula $X = \frac{\partial V}{\partial \xi}$, valid for free space, requires examination before we can be sure that it is true at a point in the substance of the body. And if it be objected that the formula is known to be true in all cases of physical interest, and that no such interest attaches to its validity or otherwise in the case which has avowedly no physical significance, an answer is that if we decide to use a certain kind of mathematical functions as approximate representations of physical quantities, we must become acquainted with the meanings and properties of these functions before we can make intelligent use of them.

Hence it is natural for the student of the theory of attractions to turn his attention to that part of pure mathematics which has to do with the definition and properties of volume and surface integrals.

III. Volume integrals.

10. Let f be a function of position, and let a finite volume T be divided into a great number of elements $\Delta\tau$, of small linear dimensions; let f_1 be a quantity associated with an element of volume $\Delta\tau$, chosen according to some law, so that it is either the value of f at some point of the element, or at any rate not greater than the greatest or less than the least value of f for points in the element. If f is finite at all points in the volume T, the sum $\Sigma f_1 \Delta\tau$ extended to all elements of T

is finite, and will remain so no matter how small and correspondingly numerous are the elements $\Delta\tau$. If this sum tends to a limit as the number of elements tends to infinity, and the linear dimensions of each tend to zero, and if this limit is independent of the law specifying f_1 and of the manner of subdivision into elements, the limit is called the volume integral of f through the volume T, and is denoted by $\int f d\tau$. This definition is only valid on the supposition that f is finite at all points in T.

Whether the limit here spoken of does or does not exist depends on the nature of the function f; we shall assume that it does exist for all the forms of f which we meet with in potential theory.

11. Next consider the case in which f is a function which becomes infinite at a point P within the volume T; clearly we need a new definition, and that which has been generally adopted is as follows. Surround the point P by a small closed surface t, and take the volume integral through the whole of the volume T except the part included by t; we thus exclude P from the range of integration, and so get a finite integral. Now let the surface t become smaller and smaller, whilst always surrounding P; if the volume integral tends to a finite limit as the space enclosed by t tends to vanishing, and if the limit is independent of the shape of t, then this limit is defined to be the integral of f throughout the whole volume T. The definition may be expressed symbolically thus:

$$\int^T f d\tau \equiv \operatorname*{Lim}_{t \to 0} \int_t^T f d\tau,$$

where the symbol $\to$ is used to denote such phrases as 'tending towards' or 'tends towards,' so that $t \to 0$ reads 'as t tends towards zero.' Here and elsewhere the subscript to the integral specifies the inner boundary of the region of integration.

If we call the space inside the vanishing surface t a 'cavity' in the volume of integration, we see at once the parallelism between the definition of this kind of volume integral and that of the so-called potential and attractions at a point in the substance of a continuous body.

The volume integral (if it exists) through a region within which f becomes infinite at some point is seen, by the above definition, to be a mathematical conception of a different character from the integral for a region in which f is everywhere finite. In a sense one might say that the latter is a true volume integral while the former is the limit of a true volume integral. The latter bears to the former the kind of

relation that a single limit bears to a double limit, or that a finite series bears to the so-called sum of an infinite series.

12. Analogously with the terminology of series, we speak of the volume integral as convergent if it tends to a finite limit with the vanishing of the cavity, divergent if it tends to become infinitely great, and semi-convergent if, as sometimes happens, there is a finite limit whose value is not independent of the shape of the cavity. Divergent integrals are, for ordinary purposes, as meaningless as divergent infinite series, and so we must satisfy ourselves that the integrals in use in gravitation problems are convergent either absolutely or in the conditional manner corresponding to semi-convergence.

To decide whether, for a given form of f, the integral is convergent or not, we have the following rule, depending on the order of the infinity of f at P in terms of r, the distance from P to the point at which f is estimated. If f becomes infinite at P of an order lower than r^{-3} the volume integral is convergent, if of an order higher than r^{-3} the integral is divergent; if of the order r^{-3} exactly the integral may be divergent, semi-convergent, or convergent, according to the way in which f in the neighbourhood of P depends on the angular position of r. This rule is not stated with sufficient accuracy to rank as a theorem, and one can easily think of exceptions to it; for example the case of $f \equiv r^{-4} \cos \theta$ (in the notation of spherical polar coordinates), which is obviously convergent for any cavity symmetrical about the plane $\theta = \frac{1}{2}\pi$ though otherwise likely to be divergent, shews that something like semi-convergence may be associated with infinities of order greater than 3; but the rule is a convenient approximation to the facts.

13. With a view to justifying the rule here given, it will be convenient to re-state, with slight modification of form, the definition of convergence. The integral of f through the volume T, which includes a point of infinity P, is convergent if, corresponding to any arbitrarily chosen small quantity σ, there can always be found a closed surface θ surrounding P such that all closed surfaces t surrounding P and lying wholly inside θ have the property that

$$\left| \int_t^\theta f d\tau \right| < \sigma.$$

That this is essentially the same as the definition of § 11 appears at once when we think of the ordinary definition of a limit; for if the integral through T has a limit A, we can choose θ so that

$$\left| \int_\theta^T f d\tau - A \right| < \tfrac{1}{2}\sigma, \quad \left| \int_t^T f d\tau - A \right| < \tfrac{1}{2}\sigma,$$

and therefore

$$\left|\int_t^\theta f d\tau\right| = \left|\int_t^T f d\tau - \int_\theta^T f d\tau\right| < \sigma;$$

it is, of course, to be understood that P must not lie on the surface θ. Thus the property constituting the definition of convergence of the present Article is a consequence of the property laid down as a definition in § 11.

Conversely, possession by an integral of the property specified in the present Article involves as a necessary consequence the existence of a limit A, though giving no indication of its actual value. For by taking θ sufficiently small we can keep the fluctuation of the value of the integral for different cavities within θ as small as we please, that is small without limit; and infinitely restricted fluctuation is the same as infinite approximation to some definite (and therefore finite) value.

Part of the rule of § 12 may be formulated in the following theorem. *If within a sphere of finite radius* (a), *having P as centre, f is everywhere less in absolute value than $Mr^{-\mu}$, where M is a (finite) constant and $\mu < 3$, the integral is convergent.* To prove this let us take for the surface θ the sphere $r = \eta$, where $\eta < a$, and let us denote by ϵ the distance from P to the nearest point of the surface t of the cavity; the cavity is of course entirely inside θ, but is otherwise unrestricted as to shape. Since the modulus of a sum is not greater than the sum of the moduli, and since an integral is the limit of a sum,

$$\begin{aligned}\left|\int_t^\theta f d\tau\right| &\leqslant \int_t^\theta |f|\, d\tau,\\ &\leqslant \int_\epsilon^\theta |f|\, d\tau,\\ &< M\int_\epsilon^\theta r^{-\mu}\, d\tau,\end{aligned}$$

where the subscript ϵ means that the inner boundary of the integrals is the sphere $r = \epsilon$; the second inequality holds because $|f|$ is positive and ϵ is completely inside θ.

In dealing with a function of r only, we may combine all elements $d\tau$ that lie between spheres of radii r and $r + dr$ in the single expression $4\pi r^2 dr$, so that

$$\begin{aligned}\left|\int_t^\theta f d\tau\right| &< 4\pi M \int_\epsilon^\eta r^{2-\mu}\, dr,\\ &< \frac{4\pi M}{3-\mu}(\eta^{3-\mu} - \epsilon^{3-\mu})\\ &< \frac{4\pi M}{3-\mu}\,\eta^{3-\mu},\end{aligned}$$

it being noted that $\epsilon < \eta$ and that $3-\mu$ is positive, so that the last expression obtained is positive. Now if σ be any arbitrarily chosen small quantity, we have only to take η less than $\{(3-\mu)\,\sigma/4\pi M\}^{\frac{1}{3-\mu}}$ in order to get a surface θ such that

$$\left|\int_t^\theta f d\tau\right| < \sigma,$$

whatever shape t may have provided only it lies inside θ. Thus the convergence of the integral of f is established.

It will be noticed that the convergence of the integral of f in accordance with this theorem involves also, as the proof indicates, the convergence of the integral of $|f|$.

It need hardly be pointed out that the position and shape of the outer boundary T of the region of integration do not, in general, affect the question of convergence; whatever the outer boundary may be, provided it does not include other points of infinity, it is only the part of the volume just round P that is in danger of making the integral very great, and so only that part need be studied with a view to detecting divergence.

It is clear that the theorem holds equally well for cases in which the point P where the infinity occurs is not inside but just on the boundary of the region of integration.

14. The corresponding theorem for divergence is as follows. *If within a sphere of finite radius* (a), *having P as centre, f is everywhere algebraically greater than $mr^{-\mu}$ where m is a constant greater than zero, and $\mu \geqslant 3$, the integral is divergent.* To prove this, we take as outer boundary the sphere $r = a$, and as inner boundary a surface t, and we denote by ϵ the distance from P to the furthest point of the surface t, so that the sphere $r = \epsilon$ completely surrounds the cavity. Then

$$\int_t^a f d\tau > m \int_t^a r^{-\mu}\, d\tau > m \int_\epsilon^a r^{-\mu}\, d\tau;$$

as before, we collect all the elements $d\tau$ between r and $r + dr$ into the expression $4\pi r^2 dr$, and so get

$$\int_t^a f d\tau > 4\pi m \int_\epsilon^a r^{2-\mu}\, dr,$$

$$> \frac{4\pi m}{\mu - 3}\{\epsilon^{-(\mu-3)} - a^{-(\mu-3)}\} \text{ or } 4\pi m\,\{\log a - \log \epsilon\},$$

according as μ is greater than or equal to 3. In either case the expression obtained tends to infinity for $\epsilon \to 0$; and so the integral, being greater than a quantity which tends to become endlessly great, is divergent.

15. The case of semi-convergence need only be illustrated by an example. Suppose that f is $r^{-3}\cos\theta$, and consider the values of the integral in the regions having a common outer boundary $r=a$, and having for inner boundaries in the first instance the sphere $r=\epsilon$, and in the second instance the sphere $r^2-r\epsilon\cos\theta=2\epsilon^2$, these being two small spheres of which the latter touches and completely surrounds the former.

The difference between the integrals over these two regions is the integral through the space between the two small spheres, which is certainly not zero so long as ϵ is different from zero, since, if we consider the volumes of the region cut off by a cone of small solid angle having the origin as vertex, the positive contribution to the integral from the frustum where $\cos\theta$ is positive is greater in absolute value than the negative contribution from the frustum where $\cos\theta$ is negative. Further, the magnitude of the integral is independent of ϵ, since if we multiply ϵ by k we can get the new region of integration by multiplying all radii vectores from P by k, and thus each element of volume $d\tau$ is multiplied by k^3; the subject of integration $r^{-3}\cos\theta$ is correspondingly multiplied by k^{-3}, and so the integral is unaltered. Thus the integral over the space between the small spheres, being finite when ϵ is not zero, has the same finite value as ϵ tends to vanishing; in other words there is a finite difference between the values of the original integral corresponding to the different cavities. It is clear, from the symmetry of f about the plane $\theta=\frac{1}{2}\pi$, that the integral is zero for the cavity whose centre is P, and therefore not zero for the cavity whose centre is not at P; but in neither case is it infinite. Thus the semi-convergence of the integral is demonstrated.

This example suggests the remark that two cavities are to be regarded as of different shapes even if they are similar, if they are not similarly situated with respect to P. The cavities considered in the example are both spheres, but since one has P at its centre while in the other P trisects a diameter, the cavities are regarded as of different shapes for purposes of the present discussion.

IV. Theorems connecting volume and surface integrals.

16. There is a well-known theorem connecting volume integrals with surface integrals taken over the boundary of the region of volume-integration. If the region be finite, if l, m, n denote the direction cosines of the normal drawn outwards from the region at

a point of the boundary B, and if ξ, η, ζ be functions having finite space differential coefficients at all points in the region,

$$\int (l\xi + m\eta + n\zeta)\, dS = \int \left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} \right) d\tau \quad \text{............(1)},$$

where dS represents an element of area of the boundary, the surface integral is taken over the complete boundary, and the volume integral through the complete volume. A proof is given in Williamson's *Integral Calculus*, Chapter XI.

It is worth while to enquire whether this theorem can be extended to the case in which there is a point P in the volume where the subject of volume-integration becomes infinite. The course which suggests itself is to surround the point P by a small closed surface σ, and to apply the original theorem to the region bounded internally by σ and externally by the surface B. The complete boundary consists of both B and σ, and so we get

$$\int_\sigma^B \left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} \right) d\tau = \int_B (l\xi + m\eta + n\zeta)\, dS + \int_\sigma (l\xi + m\eta + n\zeta)\, dS,$$

where the suffixes to the surface integrals indicate the surfaces over which they are taken. Now if the subject of integration of the volume integral is such as to make it convergent with respect to the infinity at P, the left-hand side of the equality tends to a definite limit as the dimensions of σ decrease towards zero. Consequently we get as the limiting form of the equality,

$$\int^B \left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} \right) d\tau$$

$$= \int_B (l\xi + m\eta + n\zeta)\, dS + \lim_{\sigma \to 0} \int_\sigma (l\xi + m\eta + n\zeta)\, dS \quad \text{...(2)}.$$

When the volume integral is convergent, the left side of (2) is a perfectly definite finite quantity, and hence the limit indicated on the right-hand side must exist and be independent of the shape of σ; it will exist, but have a value dependent on the shape of σ, if the volume integral is semi-convergent. In the former case it is frequently convenient to determine the value of the limit by taking some specially simple shape for σ, such as a sphere with centre at P. Generally, if the subject of volume integration is, in the neighbourhood of P, of order $r^{-\mu}$ $(\mu < 3)$, where r denotes distance from P, the subject of the surface integral under the limit sign is of order $r^{-\mu+1}$ where r equals the radius ϵ of the sphere σ; also dS is $\epsilon^2 d\omega$, where $d\omega$ is an element of solid angle, and so the surface integral is of order $\epsilon^{3-\mu}$ and tends to

the limit zero for $\epsilon \to 0$. If $\mu = 3$, the case of possible semi-convergence, the surface integral is of order ϵ^0, so far as its dependence on ϵ is concerned, and therefore may have for limit a value different from zero.

17. A generalisation, and at the same time a particular case, of the fundamental 'surface and volume integral theorem' is got by putting $\phi\xi$, $\phi\eta$, $\phi\zeta$, instead of ξ, η, ζ, where ϕ is another function of position which has finite space differential coefficients at all points in the region. The volume integral then becomes

$$\int \left\{ \frac{\partial}{\partial x}(\phi\xi) + \frac{\partial}{\partial y}(\phi\eta) + \frac{\partial}{\partial z}(\phi\zeta) \right\} d\tau,$$

so that the theorem takes the form

$$\int \phi \,.\, (l\xi + m\eta + n\zeta)\, dS - \int \phi \,.\, \left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} \right) d\tau$$
$$= \int \left(\xi \frac{\partial \phi}{\partial x} + \eta \frac{\partial \phi}{\partial y} + \zeta \frac{\partial \phi}{\partial z} \right) d\tau \text{ ......(3).}$$

Now ϕ is continuous and therefore finite throughout the whole region of integration, but let us suppose that some or all of the functions ξ, η, ζ, $\frac{\partial \xi}{\partial x}$, $\frac{\partial \eta}{\partial y}$, $\frac{\partial \zeta}{\partial z}$ become infinite at a point P in the region. If the infinities are such that both the volume integrals are convergent, it is clear that by introducing the cavity σ and making it tend to zero dimensions we get the relation

$$\int_B \phi \,.\, (l\xi + m\eta + n\zeta)\, dS + \underset{\sigma \to 0}{\mathrm{Lim}} \int_\sigma \phi \,.\, (l\xi + m\eta + n\zeta)\, dS$$
$$- \int^B \phi \,.\, \left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} \right) d\tau$$
$$= \int^B \left(\xi \frac{\partial \phi}{\partial x} + \eta \frac{\partial \phi}{\partial y} + \zeta \frac{\partial \phi}{\partial z} \right) d\tau \text{ ....(4),}$$

and that when the convergence is due to ξ, η, ζ being infinite of lower order than r^{-2} the limit of the surface integral is zero. When the volume integrals are semi-convergent, or when ξ, η, ζ are infinite of the same order as r^{-2}, the limit of the surface integral may be different from zero, possibly depending on the shape of the cavity; it would usually be convenient to take it in the ultimately equivalent form

$$\phi_P \,.\, \underset{\sigma \to 0}{\mathrm{Lim}} \int_\sigma (l\xi + m\eta + n\zeta)\, dS \text{ ....(5),}$$

where ϕ_P signifies the value of ϕ at the point P.

18. Green's Theorem is got from the 'surface and volume integral theorem' by putting

$$\xi = U\frac{\partial V}{\partial x}, \qquad \eta = U\frac{\partial V}{\partial y}, \qquad \zeta = U\frac{\partial V}{\partial z},$$

where U, V are functions of position; if we use the notation $\frac{\partial}{\partial \nu}$ for differentiation along the outward normal, and Δ for Laplace's operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, we get

$$\int U\frac{\partial V}{\partial \nu}\,dS = \int \Sigma\left\{\frac{\partial}{\partial x}\left(U\frac{\partial V}{\partial x}\right)\right\}d\tau,$$

whence

$$\int U\frac{\partial V}{\partial \nu}\,dS - \int U\Delta V d\tau = \int \Sigma\left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial x}\right)d\tau$$

$$= \int V\frac{\partial U}{\partial \nu}\,dS - \int V\Delta U d\tau \quad \dots\dots\dots\dots\dots(6)$$

by symmetry. These two equalities constitute Green's Theorem. The statement of the theorem requires modification when the region of integration is multiply connected and either U or V is many-valued; this modification is discussed in Maxwell's *Electricity* and in Lamb's *Hydrodynamics,* and need not be entered upon here.

If one of the functions, say U, together with all of its differential coefficients which occur in the formula (6), is finite throughout the region, and if V becomes infinite at a point P in the region, we isolate P in a cavity σ and make the dimensions of σ tend to zero. If the volume integrals converge, we get a pair of equalities precisely similar to the above, save that to the first member we must add $\lim_{\sigma \to 0}\int_\sigma U\frac{\partial V}{\partial \nu}\,dS$, and to the third member $\lim_{\sigma \to 0}\int_\sigma V\frac{\partial U}{\partial \nu}\,dS$.

Generally the convergence of all the volume integrals would involve that V should become infinite at P of an order lower than r^{-1}, since when V is a function of position relative to P, as is frequently the case, a space differentiation adds one to the order of the infinity so that ΔV is of an order higher by r^{-2} than V. In this case both the integrals over the surface σ are of the order of a positive power of the small length r, and their limits are zero.

The case of $V = r^{-1}$ is one of special interest in physical applications; it is also interesting mathematically because it is just the case in which semi-convergence is to be looked for. There is no semi-convergence however, for $\int U\Delta(r^{-1})\,d\tau$ is absolutely zero, since

$\Delta(r^{-1})=0$ at all points between σ and the outer boundary, and the other volume integrals are convergent because the subjects of integration are infinite of lower order than r^{-3}; and if all the terms but one of an equality are definite or convergent, that one cannot be semi-convergent. Clearly, since dS is comparable with $r^2 d\omega$, where $d\omega$ is an element of solid angle, $\int_\sigma r^{-1}\frac{\partial U}{\partial \nu}\,dS$ has a zero limit; but $\int_\sigma U\frac{\partial}{\partial \nu}(r^{-1})\,dS$ (when σ is a sphere of radius ϵ, which, in the absence of semi-convergence, may be assumed without loss of generality) has the same limit as

$$U_P\int_\sigma \frac{\partial}{\partial \nu}(r^{-1})\,r^2 d\omega \quad \text{or} \quad -U_P\int_\sigma \frac{\partial}{\partial r}(r^{-1})\,r^2 d\omega,$$

that is $U_P\int d\omega$ or $4\pi U_P$.

Thus Green's Theorem in this case gives the equalities

$$\int U\frac{\partial}{\partial \nu}(r^{-1})\,dS + 4\pi U_P = \int \Sigma\left\{\frac{\partial U}{\partial x}\frac{\partial}{\partial x}(r^{-1})\right\} d\tau$$

$$= \int r^{-1}\frac{\partial U}{\partial \nu}\,dS - \int r^{-1}\,\Delta U d\tau \quad \text{.........(7).}$$

Almost identical reasoning applies to the case in which V satisfies $\Delta V=0$ at all points of the region other than P, and becomes infinite at P in such a way that $\lim_{r\to 0}(rV)=M$, where M is a definite constant. The theorem becomes

$$\int U\frac{\partial V}{\partial \nu}\,dS + 4\pi M U_P = \int \Sigma\left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial x}\right) d\tau$$

$$= \int V\frac{\partial U}{\partial \nu}\,dS - \int V\Delta U d\tau.$$

Another case of Green's Theorem much used in Physics is that in which $U=V$. The two equalities reduce to the single one

$$\int V\frac{\partial V}{\partial \nu}\,dS - \int V\Delta V d\tau = \int \Sigma\left(\frac{\partial V}{\partial x}\right)^2 d\tau \text{............(8);}$$

no particular interest attaches to an examination of possible modifications in this formula when V has an infinity at a point in the region of integration.

V. The differentiation of volume integrals.

19. We shall next discuss the possibility of differentiating a volume integral with respect to a parameter which occurs in the subject of integration but does not affect the boundary of the region of

integration. The only parameters that need be considered here are the coordinates of a point P at which the subject of integration f has an infinity. We shall call these coordinates ξ, η, ζ, keeping x, y, z to denote the coordinates of the element $d\tau$ of integration. The question to be settled is whether the integral has a differential coefficient with respect to ξ, and if so whether that differential coefficient is equal to the integral of $\partial f/\partial\xi$. Integration means passage to a limit, and so also does differentiation; we have to find out whether alteration of the order of the two passages to limit alters the value of the final result. When P is in the region of integration there is in each case the additional passage to limit corresponding to the closing of the cavity, and the question to be settled is whether by first integrating, second closing the cavities, third making $\Delta\xi \rightarrow 0$, (where $\Delta\xi$ is an increment of ξ), we get the same result as by first making $\Delta\xi \rightarrow 0$, second integrating, third closing the cavity.

20. First we shall consider the case in which P is outside the region of integration, and shew that if f has at all points of the region and for all contemplated values of ξ a differential coefficient with respect to ξ, which differential coefficient is a uniformly continuous* function of ξ throughout the region, the differential coefficient of the integral is the same as the integral of the differential coefficient.

The incremental ratio of the integral is

$$\frac{1}{\Delta\xi}\int\{f(\xi+\Delta\xi)-f(\xi)\}\,d\tau,$$

which, by the theorem of mean value,

$$=\int f'(\xi+\theta\Delta\xi)\,d\tau$$
$$=\int f'(\xi)\,d\tau+\int\epsilon d\tau,$$

where $\quad \epsilon\equiv f'(\xi+\theta\Delta\xi)-f'(\xi)$, and $1>\theta>0$.

Now the uniform continuity of $f'(\xi)$ implies that for an arbitrarily chosen small quantity σ we can always find a quantity ω such that for all values of $\Delta\xi$ less than ω,

$$|f'(\xi+\Delta\xi)-f'(\xi)|,$$

and consequently also $|\epsilon|$, is less than σ, for all points in the region T of integration. Thus by choosing $\Delta\xi$ less than ω we can ensure that $|\int\epsilon d\tau|$ shall be less than σT, which, in virtue of the finiteness of T and the arbitrariness of σ, is arbitrarily small. In fact the difference between the

* It can be proved that if a function is continuous at all points in a region it is *uniformly* continuous throughout the region.

integral of $f'(\xi)$ and the incremental ratio of the integral of $f(\xi)$ can be made arbitrarily small. Hence the limit of the incremental ratio, i.e. $\frac{\partial}{\delta\xi}\int f(\xi)\,d\tau$, is equal to $\int f'(\xi)\,d\tau$, as we set out to prove.

21. Next we consider the case in which P is within the region of integration. The rough rule for this case is that if the original integral is convergent, and if the integral obtained by differentiating under the sign of integration is convergent, the latter is the differential coefficient of the former.

It will serve our purpose to prove this proposition for a particular case, namely that in which the subject of integration f is the product of two factors, each subject to special restrictions. One of these, which we denote by $\rho(x, y, z)$ or briefly by ρ, is supposed to be a function of absolute position, not involving ξ, η, ζ at all; it is assumed to be finite, not to vanish at P, and to have space differential coefficients which are uniformly continuous throughout the region of integration. The other factor is supposed to be a function of position relative to P, and to become infinite at P; we may denote it by $\phi(x-\xi, y-\eta, z-\zeta)$, or sometimes for brevity by $\phi(\xi, x)$ or $\phi(\xi)$. It has obviously the property that $\phi(\xi + \Delta\xi, x) = \phi(\xi, x - \Delta\xi)$, so that

$$\frac{\partial\phi}{\partial\xi} = -\frac{\partial\phi}{\partial x}.$$

The integral to be differentiated is $\int^T \rho\,.\,\phi\,.\,d\tau$, or, written in full, $\operatorname*{Lim}_{\epsilon\to 0}\int_\epsilon^T \rho\,.\,\phi\,.\,d\tau$, where ϵ is a cavity surrounding the point P. The incremental ratio is

$$\frac{1}{\Delta\xi}\left[\operatorname*{Lim}_{\epsilon'\to 0}\int_{\epsilon'}^T \rho(x, y, z)\,.\,\phi(\xi + \Delta\xi)\,.\,d\tau - \operatorname*{Lim}_{\epsilon\to 0}\int_\epsilon^T \rho(x, y, z)\,.\,\phi(\xi)\,d\tau\right],$$

where ϵ' is a cavity surrounding the point P' $(\xi + \Delta\xi, \eta, \zeta)$, which may be taken to be in all respects similar to ϵ. To get the differential coefficient of the integral it is necessary, in the incremental ratio, first to make ϵ and ϵ' tend to vanishing, and afterwards to make $\Delta\xi \to 0$.

Before we pass to either limit, however, we are at liberty to simplify the form of the incremental ratio in any manner that seems desirable. In the integral $\int_{\epsilon'}^T \rho(x, y, z)\,.\,\phi(\xi + \Delta\xi)\,.\,d\tau$ imagine the boundaries of the region of integration, and every volume element, shifted a distance $\Delta\xi$ in the negative direction of the axis of x. An element of volume originally at a point K' is merely shifted to a point K whose position

relative to the point (ξ, η, ζ) is the same as that of K' relative to $(\xi + \Delta\xi, \eta, \zeta)$; so that if K is (x, y, z), the value of ϕ belonging to the shifted volume element, originally presenting itself as $\phi(x + \Delta\xi, \xi + \Delta\xi)$, is equally well represented by the form $\phi(x, \xi)$; but the value of ρ for the element now brought to K is that appropriate to K', i.e. $\rho(x + \Delta\xi, y, z)$. The inner boundary ϵ' of the integral is brought by the shifting into coincidence with ϵ, but the outer boundary is changed to a surface T' which is simply T displaced without change of form. In fact

$$\int_{\epsilon'}^{T} \rho(x) \cdot \phi(\xi + \Delta\xi, x)\, d\tau = \int_{\epsilon}^{T'} \rho(x + \Delta\xi) \cdot \phi(\xi, x)\, d\tau,$$

so that the incremental ratio of the integral with as yet unclosed cavity is equal to

$$\frac{1}{\Delta\xi}\left[\int_{\epsilon}^{T'} \rho(x + \Delta\xi) \cdot \phi(\xi)\, d\tau - \int_{\epsilon}^{T} \rho(x) \cdot \phi(\xi) \cdot d\tau\right]$$

or to

$$\frac{1}{\Delta\xi}\left[\int_{\epsilon}^{T} \{\rho(x + \Delta\xi, y, z) - \rho(x, y, z)\}\, \phi(\xi)\, d\tau + \int_{T}^{T'} \rho(x + \Delta\xi, y, z) \cdot \phi(\xi)\, d\tau\right]$$

where the latter integral is extended to the region between T and T', being taken positively where the boundary of T' lies outside T, negatively where the reverse is the case. By the theorem of mean value the above expression equals

$$\int_{\epsilon}^{T} \rho_x'(x + \theta\Delta\xi, y, z) \cdot \phi(\xi) \cdot d\tau + \frac{1}{\Delta\xi}\int_{T}^{T'} \rho(x + \Delta\xi) \cdot \phi(\xi) \cdot d\tau,$$

$$= \int_{\epsilon}^{T} \rho_x'(x, y, z) \cdot \phi(\xi) \cdot d\tau + \int_{\epsilon}^{T} \omega\, \phi(\xi)\, d\tau + \frac{1}{\Delta\xi}\int_{T}^{T'} \rho(x + \Delta\xi) \cdot \phi(\xi) \cdot d\tau,$$

where $\omega \equiv \rho_x'(x + \theta\Delta\xi, y, z) - \rho_x'(x, y, z)$, and $1 > \theta > 0$.

Now as P is not on the boundary of T, $\Delta\xi$ can always be taken small enough to prevent P from being in the region between T and T'; hence the integral for this region has no dependence on the cavity ϵ. The assumed uniform continuity (which includes finiteness) of ρ_x', and the assumed convergence of the original integral, are sufficient guarantees of the convergence of the integrals of $\rho_x'\phi$ and $\omega\phi$. Hence we may proceed to the limits for $\epsilon \to 0$ and $\epsilon' \to 0$, and so get the relation:

Incremental ratio of $\int^{T} \rho\phi\, d\tau$

$$= \int^{T} \rho_x'\phi\, d\tau + \int^{T} \omega\phi\, d\tau + \frac{1}{\Delta\xi}\int_{T}^{T'} \rho(x + \Delta\xi) \cdot \phi(\xi) \cdot d\tau \ldots (9),$$

and the cavities ϵ, ϵ' are now closed up and finished with.

As $\Delta\xi$ becomes smaller it is clear that the volume between T and T' approximates to a very thin shell over the surface of T whose normal thickness (outwards from T) is $-\Delta\xi \,.\, l$, where l is the x cosine of the outward normal. Thus the corresponding volume integral approximates to and has the same limit as the surface integral

$$-\Delta\xi \int_T l\rho\phi\, dS.$$

And in virtue of the uniform continuity of ρ_x', corresponding to any arbitrary small quantity σ, we can always find a quantity κ such that for all values of $\Delta\xi$ less than κ, and for all points of the region of integration, $|\omega| < \sigma$, and so

$$\left| \int^T \omega\phi\, d\tau \right| < \sigma \int^T |\phi|\, d\tau.$$

This last expression is σ multiplied by a finite quantity, for, since ρ is finite and does not vanish at P, the convergence of $\int^T |\phi| d\tau$ is implied in the assumed convergence of $\int^T \rho\phi\, d\tau$, at least if the latter be convergence of the kind discussed in § 13. Hence $\int^T \omega\phi\, d\tau$ can, by suitable choice of $\Delta\xi$, be made smaller than any arbitrary small quantity, and so tends to the limit zero for $\Delta\xi \to 0$. Proceeding now to the limit $\Delta\xi \to 0$ in relation (9), we get

$$\frac{\partial}{\partial\xi} \int^T \rho\phi\, d\tau = \int^T \rho_x' \phi\, d\tau - \int_T l\rho\phi\, dS \quad \text{............(10)}.$$

The theorem of § 17, formula (4), enables us to transform the right-hand side of (10), giving

$$\begin{aligned} \frac{\partial}{\partial\xi} \int^T \rho\phi\, d\tau &= - \int^T \rho\phi_x'\, d\tau \\ &= \int^T \rho \frac{\partial\phi}{\partial\xi}\, d\tau \\ &= \int^T \frac{\partial}{\partial\xi} (\rho\phi)\, d\tau \quad \text{.................(11)}, \end{aligned}$$

provided the last integral is convergent.

If the function ρ is of such a simple character near P that the nature of the integrals depends entirely on the form of ϕ, it is clear that the case of possible semi-convergence is covered by the above reasoning, certainly as far as formula (10); but in formula (11) the

use of formula (4) may introduce an extra term on the right-hand side, namely the limit of the surface integral of $l\rho\phi$ over a new cavity round P. One can imagine that peculiarities in the form of ρ might invalidate some of the steps of the argument, but such peculiarities are not to be expected in physical applications.

VI. Applications to Potential Theory.

22. The potential at a point P (ξ, η, ζ) of a finite mass of continuous matter, whose density at a point (x, y, z) is ρ, a function of x, y, z but not of ξ, η, ζ, is the volume integral $V \equiv \int \rho r^{-1} d\tau$, where $r = \sqrt{\Sigma (x-\xi)^2}$; the attraction component parallel to the axis of x is X where $X \equiv \int (x-\xi) \rho r^{-3}\, d\tau$; both integrals are taken through the whole space occupied by the body.

When (ξ, η, ζ) is outside the body, and ρ is finite and subject to such restrictions as are required for the validity of the theorems proved in the preceding Articles, there is no infinity of the subjects of integration in the region, and so

$$X = \int \frac{\partial}{\partial \xi} (\rho r^{-1})\, d\tau = \frac{\partial}{\partial \xi} \int \rho r^{-1}\, d\tau = \frac{\partial V}{\partial \xi}.$$

Also
$$\frac{\partial^2 V}{\partial \xi^2} = \frac{\partial}{\partial \xi} \int \frac{\partial}{\partial \xi} (\rho r^{-1})\, d\tau = \int \frac{\partial^2}{\partial \xi^2} (\rho r^{-1})\, d\tau$$
$$= \int \rho \frac{\partial^2}{\partial \xi^2} (r^{-1})\, d\tau,$$

so that
$$\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} + \frac{\partial^2 V}{\partial \zeta^2} = \int \rho \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \zeta^2} \right) (r^{-1})\, d\tau = 0.$$

23. When the point P is inside the body, the potential and the attraction components have no longer the simple physical interpretation suggested by their names, but are defined as the limits of these physical quantities in a vanishing cavity. And this, as we saw in § 11, implies their equivalence to the integrals represented by the same symbols as in § 22, but referring to a region including the point of infinity P of the subjects of integration, and therefore only intelligible when the integrals are convergent.

The subject of the potential integral, being infinite at P of the order of r^{-1}, the integral is convergent; and the attraction integrals, having subjects of integration that are infinite of the order of r^{-2}, are also convergent. Hence we have, by § 21,

$$X = \frac{\partial V}{\partial \xi}.$$

But if we differentiate X with respect to ξ under the sign of integration, we get an integral whose subject of integration is of the order of r^{-3}, so that there is a possibility of semi-convergence or divergence. Instead, therefore, of merely quoting a simple differentiation rule in this case, we must proceed with care.

It is clear that for the integral $X \equiv \int \rho (x-\xi) r^{-3} d\tau$ the argument of § 21 holds as far as the formula (10), which in this case becomes

$$\frac{\partial X}{\partial \xi} = \int^T \frac{\partial \rho}{\partial x} (x-\xi) r^{-3} d\tau - \int_T l\rho (x-\xi) r^{-3} dS \quad \text{......(12)},$$

the volume integrals involved being convergent and all cavities being closed up. This formula shews that $\frac{\partial X}{\partial \xi}$ has a definite value.

Now surround P by a small surface σ and use formula (4) of § 17, putting ρ for the quantity there called ϕ, and $(x-\xi) r^{-3}$ for the quantity there called ξ. Thus we get

$$\int_T l\rho (x-\xi) r^{-3} dS + \rho_P \lim_{\sigma \to 0} \int_\sigma l (x-\xi) r^{-3} dS$$
$$- \lim_{\sigma \to 0} \int_\sigma^T \rho \frac{\partial}{\partial x} \{(x-\xi) r^{-3}\} d\tau = \int^T \frac{\partial \rho}{\partial x} (x-\xi) r^{-3} d\tau,$$

whence

$$\frac{\partial X}{\partial \xi} = \rho_P \lim_{\sigma \to 0} \int_\sigma l (x-\xi) r^{-3} dS - \lim_{\sigma \to 0} \int_\sigma^T \rho \frac{\partial}{\partial x} \{(x-\xi) r^{-3}\} d\tau \quad \text{...(13)}.$$

The sum of the limits here indicated is perfectly definite and independent of the shape of σ, but either limit taken separately has a value which depends on the shape of σ; this can be seen readily by studying the surface integral first when σ is a sphere and second when σ is a very flat cylinder with plane ends parallel to the plane of x.

Since formulae corresponding to (13) hold for Y and Z, we get by addition

$$\frac{\partial X}{\partial \xi} + \frac{\partial Y}{\partial \eta} + \frac{\partial Z}{\partial \zeta} = \rho_P \lim_{\sigma \to 0} \int_\sigma \Sigma l (x-\xi) r^{-3} dS$$
$$- \lim_{\sigma \to 0} \int_\sigma^T \rho \Sigma \frac{\partial}{\partial x} \{(x-\xi) r^{-3}\} d\tau.$$

Now here the subject of volume integration is identically zero at all points outside the cavity, and so the integral is zero whatever the shape of the cavity, and its limit is zero. Hence the value of the surface integral is independent of the shape of the cavity, and may be

calculated on the assumption that σ is the sphere $r = \epsilon$; in this case $l = -(x - \xi)\,\epsilon^{-1}$, so that $\Sigma l\,(x - \xi) = -\epsilon$, and $dS = \epsilon^2 d\omega$, where $d\omega$ is an element of solid angle at P; thus the integral becomes $-\int d\omega$, which equals -4π. Thus our equality becomes

$$\frac{\partial X}{\partial \xi} + \frac{\partial Y}{\partial \eta} + \frac{\partial Z}{\partial \zeta} = -4\pi\rho_P \quad \ldots\ldots\ldots\ldots\ldots\ldots(13),$$

or

$$\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} + \frac{\partial^2 V}{\partial \zeta^2} = -4\pi\rho_P,$$

which is Poisson's equation.

24. The theorems proved above for the differential coefficients of V are perfectly intelligible for a body of the hypothetical continuous structure which we have postulated. But when applied to a body of molecular structure such a symbol as $\frac{\partial V}{\partial \xi}$ requires qualification. It has been seen that for a continuous body $\Delta \xi$ was only made to tend to zero after we had first closed the cavities ϵ and ϵ' corresponding to V and $V + \Delta V$; in fact $\Delta \xi$, though small, was always large compared with the dimensions of the cavities, but this fact did not interfere with our making $\Delta \xi$ as near to zero as we pleased.

But for a body of molecular structure the cavities must always be large enough to be capable of containing a great number of molecules, and so we can never close them entirely; hence $\Delta \xi$, so far from ever vanishing, must be actually large compared with the smallest length which can be regarded as only physically small; nevertheless $\Delta \xi$ may be, to our senses, extremely small. Hence instead of the true differential coefficient $\frac{\partial V}{\partial \xi}$ we have the incremental ratio $\frac{\Delta V}{\Delta \xi}$, where $\Delta \xi$ though very small is still definitely prevented from attaining the higher orders of smallness which lie on the way to the limit zero. Thus it is clear that the symbol $\frac{\partial V}{\partial \xi}$, just as V itself, is inexact and stands for something not precisely defined; but the inaccuracy or deviation from a precise value is no greater than the inaccuracy which regards matter as continuous, and is in fact an inaccuracy so small as to be inappreciable to our senses. Accordingly the relations

$$X = \frac{\partial V}{\partial \xi} \quad \text{and} \quad \Sigma \frac{\partial X}{\partial \xi} = -4\pi\rho$$

have a sufficiently precise meaning when applied to bodies of molecular structure.

VII. Applications to Theory of Magnetism.

25. If a body is magnetised so that the components of the intensity of magnetisation at a point (x, y, z) are A, B, C, the magnetic potential at an external point P, (ξ, η, ζ), is given by

$$V = \int \left(A \frac{\partial}{\partial x} + B \frac{\partial}{\partial y} + C \frac{\partial}{\partial z} \right) (r^{-1})\, d\tau \quad \text{............(14)},$$

and the x component of magnetic force is α where

$$\alpha = -\int \frac{\partial}{\partial \xi} \left(A \frac{\partial}{\partial x} + B \frac{\partial}{\partial y} + C \frac{\partial}{\partial z} \right) (r^{-1})\, d\tau \quad \text{.........(15)},$$

the body being regarded as of continuous structure; so long as P is outside the body it is clear that

$$\alpha = -\frac{\partial V}{\partial \xi} \quad \text{............................(16)}.$$

Outside the body the induction (a, b, c) is the same as the force (α, β, γ), and therefore remembering that A, B, C are functions of x, y, z but not of ξ, η, ζ, while r depends only on $x-\xi$, $y-\eta$, $z-\zeta$, we see that

$$\begin{aligned}
a &= -\int \left(A \frac{\partial}{\partial x} + B \frac{\partial}{\partial y} + C \frac{\partial}{\partial z} \right) \frac{\partial}{\partial \xi} (r^{-1})\, d\tau \\
&= \int \left(A \frac{\partial^2}{\partial x^2} + B \frac{\partial^2}{\partial x \partial y} + C \frac{\partial^2}{\partial x \partial z} \right) (r^{-1})\, d\tau \\
&= \int \left(-A \frac{\partial^2}{\partial y^2} - A \frac{\partial^2}{\partial z^2} + B \frac{\partial^2}{\partial x \partial y} + C \frac{\partial^2}{\partial x \partial z} \right) (r^{-1})\, d\tau \\
&= \int \left(C \frac{\partial}{\partial x} - A \frac{\partial}{\partial z} \right) \frac{\partial}{\partial z} (r^{-1})\, d\tau - \int \left(A \frac{\partial}{\partial y} - B \frac{\partial}{\partial x} \right) \frac{\partial}{\partial y} (r^{-1})\, d\tau \\
&= -\int \left(C \frac{\partial}{\partial x} - A \frac{\partial}{\partial z} \right) \frac{\partial}{\partial \zeta} (r^{-1})\, d\tau + \int \left(A \frac{\partial}{\partial y} - B \frac{\partial}{\partial x} \right) \frac{\partial}{\partial \eta} (r^{-1})\, d\tau \\
&= \frac{\partial H}{\partial \eta} - \frac{\partial G}{\partial \zeta} \quad \text{...(17)},
\end{aligned}$$

where

$$F = \int \left(B \frac{\partial}{\partial z} - C \frac{\partial}{\partial y} \right) (r^{-1})\, d\tau, \qquad G = \int \left(C \frac{\partial}{\partial x} - A \frac{\partial}{\partial z} \right) (r^{-1})\, d\tau,$$

$$H = \int \left(A \frac{\partial}{\partial y} - B \frac{\partial}{\partial x} \right) (r^{-1})\, d\tau \quad \text{..................(18)}.$$

F, G, H are the components of the vector-potential at P; the relation between induction and vector potential is frequently written

$$(a, b, c) = \text{curl}\,(F, G, H) \quad \text{.....................(19)}.$$

26. When P is inside the magnetised body the integral of formula (14) is convergent, and so the formula may stand as the definition of the potential at P. The properties of V, thus defined, are most easily deduced from another expression, obtained by making a cavity round P and applying the theorem of § 17, formula (4), taking ϕ to be r^{-1} and writing A, B, C for ξ, η, ζ. The surface integral over the cavity has a zero limit, and so we get

$$V = \int_T (lA + mB + nC)\, r^{-1}\, dS - \int^T \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right) r^{-1}\, d\tau \ldots(20),$$

where the surface integral refers only to the boundary T of the body, and the cavity is now closed and finished with. This form of the magnetic potential exhibits it as equivalent to the *gravitation* potential of a volume distribution of density

$$\rho = -\frac{\partial A}{\partial x} - \frac{\partial B}{\partial y} - \frac{\partial C}{\partial z},$$

combined with a distribution of surface density $lA + mB + nC$ spread over the boundary of the body. (Of course formula (20) is equally true when P is outside the body.)

If we suppose that P is right inside the body, i.e. not on the boundary, there is no infinity in the subject of surface integration; the volume-integral part of V has the properties of the gravitation potential studied in Section VI. Thus V has definite space differential coefficients obtained by differentiating under the sign of integration in formula (20) (not in formula (14)*); accordingly, since

$$\frac{\partial}{\partial \xi}(r^{-1}) = -\frac{\partial}{\partial x}(r^{-1}),$$

$$-\frac{\partial V}{\partial \xi} = \int_T (lA + mB + nC)\frac{\partial}{\partial x}(r^{-1})\, dS - \int^T \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right)\frac{\partial}{\partial x}(r^{-1})\, d\tau,$$

the volume integral having a perfectly definite value. Now we cut a cavity σ round P, and apply the theorem of § 17, formula (4), and we get

$$-\frac{\partial V}{\partial \xi} = -\operatorname*{Lim}_{\sigma \to 0}\int_\sigma (lA + mB + nC)\frac{\partial}{\partial x}(r^{-1})\, dS$$
$$+ \operatorname*{Lim}_{\sigma \to 0}\int_\sigma^T \left(A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} + C\frac{\partial}{\partial z}\right)\frac{\partial}{\partial x}(r^{-1})\, d\tau \ldots(21).$$

These two limits combined give a value which is independent of the shape of σ, but the value of each limit taken separately depends on the

* The theorem of § 21 does not apply to the integral of formula (14), since the infinity at P is of the order r^{-2}.

shape of σ; the volume integral we see, by comparison with (15), to be the x component of the magnetic force in the cavity due to all the matter outside the cavity. So in general $-\frac{\partial V}{\partial \xi}$ is not the limit of the component of force in the cavity, but differs from it by an amount represented by the limit of the surface integral. If, however, we can choose a shape for the cavity which shall make the limit of the surface integral zero, the limit of the force component in the cavity will be accurately represented by $-\frac{\partial V}{\partial \xi}$; and this is effected by making the cavity a cylinder whose generators are parallel to the direction of the vector (A, B, C) at the point P, with flat ends perpendicular to the generators, all the linear dimensions of the cylinder tending to zero in such fashion that the linear dimensions of the ends tend to become vanishingly small compared with the length; this may, for brevity, be called a 'long' cylinder. The direction chosen for the generators ensures that the integral of $lA + mB + nC$ for the curved portion of the surface tends to zero, and the relative smallness of the flat ends makes the integral over these tend also to zero. The *definition* of the magnetic force (α, β, γ) at a point P in the body is 'the limit of the force in a cavity in the form of a long cylinder with generators parallel to the resultant intensity of magnetisation'; and this definition, in connexion with the present argument, justifies the statement that

$$\alpha = -\frac{\partial V}{\partial \xi}.$$

The *definition* of the induction (a, b, c) at a point P in the body is 'the limit of the force in a cavity in the form of a very flat circular cylinder with generators parallel to the resultant intensity of magnetisation,' where by a very flat cylinder is meant one whose linear dimensions tend to zero in such a way that the length tends to become vanishingly small in comparison with the linear dimensions of the plane ends. For such a cavity $lA + mB + nC$ tends to zero on the curved part of the surface, to $-I$ over one of the plane ends, and to $+I$ over the other, I being the resultant intensity of magnetisation; and each of these ends ultimately subtends a solid angle 2π at P. Thus the three surface integrals of which that in (21) is a type have for limits the components of force at a point between two infinite circular* parallel planes, the

* The word 'circular' is introduced in order to exclude cases in which the resultant force at a point between the parallel planes is not normal to them. The circles are supposed to have a common axis, passing through P.

one covered with a uniform surface density I, the other with a uniform surface density $-I$, of matter that attracts according to the Newtonian law; this force is known to be $4\pi I$ perpendicular to the planes, and so its components are $4\pi A$, $4\pi B$, $4\pi C$. So, for the flat cavity, (21) yields the equality

$$\alpha = -4\pi A + a \qquad \text{.........................(22).}$$

27. The integrals of formulae (18) representing vector potential are convergent for a point inside the body, and may therefore stand as the definition of the vector potential at such a point. If in the formula (4) of § 17 we put 0 for ξ, $-C$ for η, B for ζ, and r^{-1} for ϕ, we get

$$F = \int_T (nB - mC)\, r^{-1} dS - \int^T \left(\frac{\partial B}{\partial z} - \frac{\partial C}{\partial y}\right) r^{-1}\, d\tau \quad \text{......(23),}$$

the cavity of formula (4) being closed and finished with, and the surface integral over the cavity having a zero limit. This result exhibits F as the gravitation potential of a finite volume distribution combined with a surface distribution; it shews, therefore, that F has definite differential coefficients with respect to the coordinates of P.

From the two formulae analogous to (23),

$$\frac{\partial H}{\partial \eta} - \frac{\partial G}{\partial \zeta} = \int_T \left[(Cl - An)\frac{\partial}{\partial z} - (Am - Bl)\frac{\partial}{\partial y}\right] r^{-1}\, dS$$
$$+ \int^T \left[\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right)\frac{\partial}{\partial y} - \left(\frac{\partial C}{\partial x} - \frac{\partial A}{\partial z}\right)\frac{\partial}{\partial z}\right] r^{-1}\, d\tau \text{...(24).}$$

Cut a cavity σ round P and apply the theorem of formula (4) § 17, and the result is readily seen to be

$$\frac{\partial H}{\partial \eta} - \frac{\partial G}{\partial \zeta} = -\lim_{\sigma \to 0} \int_\sigma \left[(Cl - An)\frac{\partial}{\partial z} - (Am - Bl)\frac{\partial}{\partial y}\right] r^{-1}\, dS$$
$$+ \lim_{\sigma \to 0} \int_\sigma^T \left[\left(C\frac{\partial}{\partial x} - A\frac{\partial}{\partial z}\right)\frac{\partial}{\partial z} - \left(A\frac{\partial}{\partial y} - B\frac{\partial}{\partial x}\right)\frac{\partial}{\partial y}\right] r^{-1}\, d\tau \text{...(25),}$$

wherein the limits on the right-hand side together give a value independent of the shape of σ, though the value of each separately depends on the shape of σ. The volume integral is the same as

$$\int_\sigma^T \left(A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} + C\frac{\partial}{\partial z}\right)\frac{\partial}{\partial x}(r^{-1})\, d\tau$$

and accordingly represents the x component of force due to all the magnetisation outside the cavity; the surface integral, together with the corresponding surface integrals in the two other formulae analogous to (25), will tend to zero if $A/l = B/m = C/n$ over practically the whole

surface of the cavity, and this is ultimately the case when the cavity is the flat cylinder used in defining the induction. For this shape of cavity (25) is equivalent to

$$\frac{\partial H}{\partial \eta} - \frac{\partial G}{\partial \zeta} = a \quad\text{..........................(26)},$$

which, with the two other equalities of the same type, constitutes the vector relation

$$(a, b, c) = \text{curl}\,(F, G, H),$$

true now for points inside as well as for points outside the magnetised body.

It should be noticed that the definition of vector potential used in the present discussion is not that which is regarded as fundamental in the physical theory, though equivalent to it. The usual definition is contained in the relation (19) coupled with the relation

$$\frac{\partial F}{\partial \xi} + \frac{\partial G}{\partial \eta} + \frac{\partial H}{\partial \zeta} = 0.$$

It is easy to verify, on the lines of the present Article, that the vector defined by the relation (18), whether for internal or for external points, satisfies this further condition.

VIII. Surface Integrals.

28. When gravitating matter is distributed in a very thin layer, or when the surface of a body is charged with electricity, the corresponding potential and attraction at a point P are represented by surface integrals, a surface density σ taking the place of a volume density. The integrals are of the type $\int \sigma f dS$ where σ is usually free from such mathematical peculiarities as might raise doubts concerning the existence of the integrals, and f is a function having an infinity at the point P (ξ, η, ζ).

So long as P is not actually in the region of integration the integrals do not present any difficulties, and the formulae $X = \frac{\partial V}{\partial \xi}$, $\Delta V = 0$, are clearly valid.

When the point P is in the surface distribution we must cut a cavity round it of dimensions that tend to zero, and the question of convergence necessarily arises. We consider first the case in which the integration takes place over a portion of a *plane* surface.

The chief test of convergence is now as follows. *If within a circle of finite radius* (a), *having the point* P *as centre, the subject* ϕ *of*

integration is always less in absolute value than $Mr^{-\mu}$ where $\mu<2$ and M is a definite constant, the integral $\int\phi dS$ is convergent. To prove this we shall shew that, corresponding to any arbitrarily chosen small quantity σ, there can always be found a closed curve θ surrounding P such that all closed curves t surrounding P and lying wholly inside θ have the property that

$$\left|\int_t^\theta \phi dS\right| < \sigma.$$

Take for the curve θ the circle $r=\eta$, where $\eta<a$, and denote by ϵ the distance from P to the nearest point of the boundary t of the cavity; the cavity is of course entirely inside θ, but is otherwise unrestricted as to shape. Since the modulus of a sum is not greater than the sum of the moduli,

$$\left|\int_t^\theta \phi dS\right| \leqslant \int_t^\theta |\phi|\, dS,$$

$$\leqslant \int_\epsilon^\theta |\phi|\, dS, \quad < M\int_\epsilon^\theta r^{-\mu}\, dS,$$

$$< 2\pi M\int_\epsilon^\eta r^{1-\mu}\, dr, \quad < \frac{2\pi M}{2-\mu}(\eta^{2-\mu}-\epsilon^{2-\mu}), \quad < \frac{2\pi M}{2-\mu}\eta^{2-\mu}.$$

Hence by choosing η less than $\{(2-\mu)\sigma/2\pi M\}^{\frac{1}{2-\mu}}$ we get a curve θ satisfying the specified condition; the integral is accordingly convergent.

When the order of the infinity of ϕ is the same as that of r^{-2}, semi-convergence may appear.

29. Passing to the case in which the region of integration is a portion of a *curved* surface, we shall assume P to be a point at which there is a definite tangent plane and such that at all points of the region within a finite distance of P the principal curvatures are both finite. We need consider only the integral taken through a finite region not extending far from P, and in virtue of the finiteness of the curvatures at and near P we can always choose this region so that, if Q is any point of it and θ the inclination of the tangent plane at Q to the tangent plane at P, for all positions of Q in the region $\theta<\alpha$, where α is a definite acute angle. Let the projection of Q on the tangent plane at P be Q_0, let r, r_0 represent PQ, PQ_0 respectively, dS an element of area round Q, dS_0 the projection of dS on the tangent plane at P, B the boundary of the area of integration, and B_0 its projection on the tangent plane at P.

Since $dS = dS_0 \sec\theta$,

$$\int^B \phi\, dS = \int^{B_0} \phi \sec\theta\, dS_0,$$

the second integral being taken in the tangent plane at P.

If within the region of integration

$$|\phi| < Mr^{-\mu}$$

where M is a constant and $2 > \mu > 0$, then

$$|\phi \sec\theta| < Mr^{-\mu} \sec\theta$$
$$< Mr_0^{-\mu} (r_0/r)^\mu \sec\theta,$$

or, since $r_0 < r$, and $\sec\theta < \sec\alpha$,

$$|\phi \sec\theta| < M \sec\alpha\, r_0^{-\mu},$$

where $M \sec\alpha$ is finite since α is acute.

Hence $\int^{B_0} \phi \sec\theta\, dS_0$ is convergent, and therefore so also is $\int^B \phi\, dS$; thus the test of convergence is the same whether the surface of integration be plane or curved provided the curvatures be finite. The existence of a definite tangent plane at P is not a necessary feature in the proof, the essential thing is that there shall be a finite region round P for which $\theta < \alpha < \frac{1}{2}\pi$, θ being inclination to some fixed plane through P; for example the surface might be a cone and P its vertex. (Compare Poincaré, *Potentiel Newtonien*, § 33.)

30. Applying the test of the preceding Articles we see that at a point in a surface distribution of gravitating matter or electricity the potential is represented by a convergent integral, but the attraction components in the tangent plane are represented by integrals whose order renders semi-convergence possible. It is not difficult to shew, by a particular example, that semi-convergence does occur; for the attraction of a uniform plane elliptic disc (of eccentricity e and surface density σ) at a focus is $2\pi\sigma(1 - \sqrt{1 - e^2})/e$ if the cavity is circular, but is zero if the cavity is an ellipse similar and similarly situated to the edge of the disc, with the focus for centre of similitude; the verification of these statements, by using polar coordinates and integrating, is quite easy.

The component of attraction at P normal to the surface is represented by a convergent integral, but this quantity is the attraction *in a cavity*, though a vanishing one, and must be distinguished from the normal component of attraction at a point very close to the *unbroken* surface but not in it; it is, in electrical applications, the

'mechanical force per unit charge,' the quantity denoted by R_2 in Prof. J. J. Thomson's *Elements of Electricity and Magnetism*, § 37, whereas the normal attraction at a point just not in the surface is the quantity there denoted by R.

31. The distinction drawn above, between the attraction at a point in the surface and that at a point just not in the surface, brings us to a question of a kind which, for lack of a fourth dimension, does not arise geometrically in the case of volume integrals, the question, namely, whether an integral $\int \phi \, dS$ tends to a definite limit if the point P, where ϕ has an infinity, is not originally in the surface, but approaches a point of the surface as a limiting position.

Let O be the point of the surface to which P gets continually nearer; it will be convenient to take O as origin of coordinates and the tangent plane at O as plane of z; we shall suppose that there is a limiting position of the line PO, as P moves up to coincidence with O, which makes with the plane of z a definite angle different from zero and so has definite direction cosines l_0, m_0, n_0, of which the last is numerically greater than zero. The length PO will be denoted by κ, and the coordinates of P by (ξ, η, ζ) or $(-l\kappa, -m\kappa, -n\kappa)$, while x, y, z represent the coordinates of a variable point Q on the surface; the subject of integration, $\phi\ (x, y, z, \xi, \eta, \zeta)$, may for brevity be represented by ϕ, while $\phi\ (x, y, z, 0, 0, 0)$, the value of ϕ when P is coincident with O, will be represented by ϕ_0. If integration be extended to a finite part of the surface round O, bounded by a closed curve B, the quantities to whose different meanings and possibly different values it is desired to draw attention are respectively

$$\int^B \phi_0 \, dS \text{ and } \operatorname*{Lim}_{\kappa \to 0} \int^B \phi \, dS.$$

The first thing to notice is that, while the integral of ϕ requires no cavity so long as κ is different from zero, which is the case at all stages in the passage to limit denoted by $\kappa \to 0$, the integral of ϕ_0 is only intelligible in terms of a cavity ϵ round the point O, though this cavity of course tends to vanishing. If, therefore, we set out to find the algebraic difference between the two quantities which form the subject of discussion (which may conveniently be denoted by D) we have

$$\begin{aligned} D &\equiv \operatorname*{Lim}_{\kappa \to 0} \int^B \phi dS - \int^B \phi_0 dS \\ &= \operatorname*{Lim}_{\kappa \to 0} \int^B \phi dS - \operatorname*{Lim}_{\epsilon \to 0} \int_\epsilon^B \phi_0 dS. \end{aligned}$$

Since the term involving the limit for $\kappa \to 0$ in no way depends upon ϵ, and the term involving the limit for $\epsilon \to 0$ in no way depends upon κ, it is a matter of indifference in what order we suppose the passages to limit to be made; accordingly we are at liberty, if we please, to make first the passage to the limit for $\kappa \to 0$, so that at any stage short of the limits we shall think of κ as extremely small compared with the linear dimensions of the cavity ϵ. The difference, then, before passage to either limit, may be put in the form

$$\int^{\epsilon} \phi dS + \int_{\epsilon}^{B} \phi dS - \int_{\epsilon}^{B} \phi_0 dS.$$

Now if we proceed first to the limit for $\kappa \to 0$, the points P and O at all stages of this passage are quite outside the region of integration of the last two integrals, and the functions ϕ and ϕ_0 are kept definitely removed from their infinite values; hence in the absence of peculiarities of ϕ other than that infinity at P which is the special subject of our investigation, we get the same limit for $\int_{\epsilon}^{B} \phi dS$ whether we first integrate and then make $\kappa \to 0$ or first make $\kappa \to 0$ and then integrate. In fact

$$\operatorname*{Lim}_{\kappa \to 0} \int_{\epsilon}^{B} \phi dS = \int_{\epsilon}^{B} \operatorname*{Lim}_{\kappa \to 0} \phi dS = \int_{\epsilon}^{B} \phi_0 dS;$$

whence

$$D = \operatorname*{Lim}_{\epsilon \to 0} \left[\operatorname*{Lim}_{\kappa \to 0} \int^{\epsilon} \phi dS + \operatorname*{Lim}_{\kappa \to 0} \int_{\epsilon}^{B} \phi dS - \int_{\epsilon}^{B} \phi_0 dS \right]$$

$$= \operatorname*{Lim}_{\epsilon \to 0} \operatorname*{Lim}_{\kappa \to 0} \int^{\epsilon} \phi dS \quad \text{...(27)},$$

the notation implying that ϵ is kept constant while $\kappa \to 0$, thus yielding a limit which is a function of ϵ, and that *afterwards* the limit of this function of ϵ is taken for $\epsilon \to 0$.

Let us suppose ϕ to be of the form $(z - \zeta)^{\lambda} r^{-\mu}$ where r denotes PQ and λ and μ are positive, and let us proceed to make a closer examination of D for this particular case. We shall assume that the principal curvatures of the surface are finite at all points in a finite region round O, and that the cavity ϵ is determined by the intersection of the surface with the narrow cylinder $x^2 + y^2 = \epsilon^2$; and we picture to ourselves a small piece of the surface, which we may call the 'cap,' bounded by this curve whose projection on the plane of z is a circle of radius ϵ and centre O, and a point P at a distance κ from O which is extremely small compared with ϵ. To begin with, we observe that there is a finite constant a such that for all points Q in the cap $|z| < as^2$, where s stands

for $\sqrt{x^2+y^2}$, the distance of Q from the axis of z; for, since the surface has finite curvature at all points of the cap, $|z|/s^2$ tends to a finite limit for any given azimuth as Q approaches O, and so is finite at all points of the cap; and the various values, being finite, have a finite superior limit a which is of the same order of magnitude as the greatest curvature of a normal section through O; thus the inequality is proved.

Consider now the curve of intersection of the surface with the cylinder $x^2+y^2=\kappa^2$; this divides the cap into two regions, an inner region whose linear dimensions are of the order of κ and therefore small in comparison with those of the cap, and an outer region comprising most of the cap, in which there is no point whose distance from O is of a higher order of smallness than κ. The integral whose limit is D can be regarded as the sum of two integrals, one over the inner region, one over the outer region, and these will be considered separately.

Taking first the outer region, and remembering the assumption $n_0 \neq 0$, we see that, while there may be points in this region for which $r<s$, there are no points in it for which s/r becomes infinite, so that there is a finite superior limit β for the values of s/r in the region; the finiteness of the curvature is a guarantee that there is a finite superior limit c (differing from unity by a quantity of the order of ϵ^2) to the secant of the inclination to the axis of z of the normal to the surface at Q, i.e. the ratio of dS to its projection dS_0 on the plane of z; and $|\zeta|/s$ is finite at all points of the region and has therefore a finite superior limit γ', so that $|\zeta|<\gamma' s$; as $|z|<as^2$, it follows that

$$|z-\zeta|\, s^{-1} < \gamma' + as < \gamma$$

where γ is a finite quantity.

Thus in the outer region

$$|\phi| = |(z-\zeta)^\lambda r^{-\mu}| < \gamma^\lambda s^\lambda \beta^\mu s^{-\mu},$$

$$dS < c\,dS_0,$$

and so

$$\left|\int \phi\, dS\right| < \gamma^\lambda \beta^\mu c \int s^{\lambda-\mu} dS_0,$$

$$< 2\pi\gamma^\lambda\beta^\mu c \int_\kappa^\epsilon s^{\lambda-\mu+1} ds,$$

$$< 2\pi\gamma^\lambda\beta^\mu c\,(\lambda-\mu+2)^{-1}\left[\epsilon^{\lambda-\mu+2}-\kappa^{\lambda-\mu+2}\right].$$

The limit of this for $\kappa \to 0$ and $\epsilon \to 0$ is zero provided $\mu - \lambda < 2$.

Taking next the inner region, we know that in it $|z| < \alpha s^2 < \alpha\kappa^2$, while ζ is of the same order of smallness as κ, so that $|z-\zeta|$ is of the same order as κ, and there must be an inequality $|z-\zeta| < g\kappa$ where g is a definite constant. Clearly there is a superior limit to the secant of the angle between the normal at Q and the line PQ, so that there is an inequality $dS < hr^2 d\omega$, where $d\omega$ represents an element of solid angle at P; and r always bears to κ a finite ratio, so that there is a double inequality $p\kappa > r > q\kappa$, where p and q are constants. Hence in the inner region

$$\int (z-\zeta)^\lambda r^{-\mu} dS < g^\lambda q^{-\mu} p^2 h \kappa^{\lambda-\mu+2} \int d\omega,$$

and the limit of this, for $\kappa \to 0$, is zero provided $\mu - \lambda < 2$.

Thus $D = 0$ for $\phi = (z-\zeta)^\lambda r^{-\mu}$ subject to $\mu - \lambda < 2$, and it is an immediate inference that $D = 0$ for $\phi = \sigma(z-\zeta)^\lambda r^{-\mu}$ subject to the same condition, if σ is a function of x, y, z which is finite throughout the region of integration. For this type of integral the condition for the vanishing of D is not the same as the condition for the convergence of the integral of ϕ_0, for, in the neighbourhood of O, z becomes small of the order of r_0^2, so that the condition of convergence of ϕ_0 is

$$\mu - 2\lambda < 2.$$

Powers of $x-\xi$, $y-\eta$ might appear in ϕ; they would count as the corresponding powers of r in applying the test just proved, though of course they would not be equivalent to powers of r if one were examining a case where something analogous to semi-convergence seemed probable.

For the potential integral $\lambda = 0$, $\mu = 1$, and therefore $D = 0$. Thus the potential at O is the same as the limit of the potential at P as P approaches O, so that V is not discontinuous at points on the surface.

32. For the attraction components $\mu - \lambda = 2$, and the test of the previous Article is not applicable so that a special investigation is required. Let us consider first a tangential component, say that whose subject of integration is $\sigma(x-\xi) r^{-3}$.

The corresponding integral of ϕ_0 is semi-convergent, and it may appear useless to investigate the difference D between the unknown limit of the integral of ϕ and the quantity of uncertain value which is the integral of ϕ_0. But the integral of ϕ is quite independent of the shape of the cavity, in fact it does not require any cavity, so we are as free as in the case of absolute convergence to give to the cavity any shape we please, so long as we attach to the integral of ϕ_0 the value

associated with that particular shape; thus the semi-convergence does not introduce any uncertainty into the meaning of D.

Let us take the same cavity as in the preceding Article, using the same notation and applying whatever parts of the reasoning remain valid for the changed form of ϕ. And let us consider what error is introduced into the subject of integration if we replace σ by σ_0, its value at O, dS by its projection dS_0 on the plane of z, and r by r' the distance from P to the projection of Q on the plane of z. The error due to the change in σ corresponds to the omission of a factor which differs from unity by a quantity of the order of smallness of s, if we assume the function σ to have no troublesome peculiarities* at O; the error due to the change in dS corresponds to the omission of a factor differing from unity by a quantity of the order of s^2; and, since $|r-r'|<|z|<as^2$, the error due to the change in r corresponds to the omission of a factor differing from unity by a quantity of the order s^2r^{-1}. So the most important terms representing error in the integral over the cap are of the order of the integrals over the cap of $(x-\xi)r^{-3}s$ and $(x-\xi)r^{-4}s^2$; for these error integrals $\lambda=0$, $\mu=1$, s in the numerator counting as equivalent to r since there is a finite superior limit to sr^{-1}, and so, by the previous Article, the error has a zero limit for $\kappa \to 0$ and $\epsilon \to 0$.

Hence, for the x component of attraction,

$$D = \operatorname*{Lim}_{\epsilon\to 0}\operatorname*{Lim}_{\kappa\to 0}\int^{\epsilon}\sigma_0(x-\xi)\,r'^{-3}dS_0,$$

where it is clear that the integral is now taken over a circular area of radius ϵ in the plane of z, and represents the component of attraction at P of a circular disc of uniform surface density σ_0. Now, before passage to the limit, P is not in such a position of symmetry that the x attraction component must vanish, but if we describe the reflexion with respect to the plane $x=\xi$ of the lesser of the two arcs into which this plane divides the circumference of the disc, we obtain a division of the disc into two areas the greater of which, on account of the symmetry of the position of P with respect to it, contributes nothing to the attraction component. The component is therefore that due to the crescent-shaped smaller area, whose mass is very nearly $4\xi\epsilon\sigma_0$ and whose nearest point is at a distance from P comparable with ϵ; thus the

* If further precision be desired, we may assume $|\sigma-\sigma_0|<Ms^m$, where M is finite and m positive. The error corresponding to this is less than the integral of $Ms^m(x-\xi)r^{-3}$, for which $\lambda=0$, $\mu=2-m$, and the limit of the error is zero. In the text m is taken to be unity.

component of attraction is of the same order of magnitude as $\sigma_0\xi\epsilon^{-1}$, which has a zero limit if we make $\kappa \to 0$ (i.e. $\xi \to 0$) before $\epsilon \to 0$. Thus D is zero, so that the x component of attraction of the whole surface at P tends to a limit, as P approaches O, equal to the corresponding component of attraction at O reckoned for a vanishing circular cavity with O as centre; the limit is the same from whichever side of the surface P moves up to O.

33. In the case of the normal attraction component Z, the subject of integration is $\sigma(z-\zeta)r^{-3}$, and the corresponding component at O is represented by a convergent integral. Thus

$$D = \underset{\epsilon \to 0}{\mathrm{Lim}}\ \underset{\kappa \to 0}{\mathrm{Lim}} \int^{\epsilon} \sigma(z-\zeta)\, r^{-3}\, dS.$$

The most important terms of the error introduced into the integral of this formula by putting σ_0 for σ, r' for r, and dS_0 for dS, are of the same order of magnitude as

$$\int^{\epsilon} \sigma_0 (z-\zeta)\, sr^{-3}\, dS^* \quad \text{or} \quad \int^{\epsilon} \sigma_0 (z-\zeta)\, s^2 r^{-4}\, dS;$$

for both of these $\lambda = 1$, $\mu = 2$, and therefore, by § 31, the error has a zero limit. Accordingly D is the limit of

$$\int^{\epsilon} \sigma_0 z r'^{-3} dS_0 - \int^{\epsilon} \sigma_0 \zeta r'^{-3} dS_0;$$

and in the former of these we notice that z is of the order of smallness of s^2, and therefore the integral of the same order as $\int^{\epsilon} \sigma_0 s^2 r'^{-3}\, dS_0$, which has $\lambda = 0$, $\mu = 1$, and therefore the limit zero. Thus

$$D = -\underset{\epsilon \to 0}{\mathrm{Lim}}\ \underset{\kappa \to 0}{\mathrm{Lim}}\ \sigma_0 \int^{\epsilon} \zeta r'^{-3} dS_0,$$

the integral being now taken over the plane circular area bounded by $r' = \epsilon$, $z = 0$. Now if $d\omega$ represent the solid angle subtended by dS_0 at P, $|\zeta|\, r'^{-3} dS_0 = d\omega$, and the integral is $\pm\int d\omega$, the sign being positive if ζ is positive, negative if ζ is negative. If we make $\kappa \to 0$ before $\epsilon \to 0$, clearly the limit of $\int d\omega$ is 2π, and so we get

$$D = \mp 2\pi\sigma_0,$$

the upper sign corresponding to ζ positive. So the limit of Z differs from the value of Z at O by $2\pi\sigma_0$, the excess of the former over the latter corresponding to an attraction $2\pi\sigma_0$ *towards* the surface; the difference between the limits of Z as P approaches O from different

* If we make the same assumption with regard to σ as in the footnote of § 32 the index of s will be m in this integral.

sides of the surface is $4\pi\sigma_0$, and the arithmetic mean of these limits is the value of Z at O.

34. The potential at P of a double sheet, or normally magnetised shell, of strength μ' at the point Q, is given by

$$V = \int \mu' r^{-2} \cos\psi \, dS,$$

where ψ is the angle between QP and the normal at Q drawn in the sense for which μ' is reckoned positive. The error introduced into the subject of integration by taking ψ at points near O to mean the angle between QP and the axis of z, and so replacing $\cos\psi$ by $-(z-\zeta) r^{-1}$, corresponds to dropping a factor which differs from unity by a quantity of the order of s^2, and the integral of this error taken over the cap is one for which, in the notation of § 31, $\lambda = 1$, $\mu = 1$, and therefore has a zero limit. Hence the potential of a double sheet has, for purpose of finding D, the same form as the integral investigated in the preceding Article; and the limit of V as P approaches O from the positive side of the sheet exceeds by $4\pi\mu_0'$ the limit as P approaches O from the negative side.

35. The potential of a surface-distribution of gravitating matter, whose surface density is free from such peculiarities as would render invalid the properties already established, has in a certain sense a space differential coefficient in any direction at any point O of the surface; this is not a differential coefficient as generally defined, since it is a limit which has different values according as the consecutive point P approaches O from one side of the surface or from the other. It is to be noticed that the existence of a differential coefficient cannot be inferred from the physical property that force equals gradient of potential, since O is a point not in free space, but in the gravitating matter.

The theorem is that

$$\lim_{OP\to 0} \{(V_O - V_P)/OP\} = \lim_{OP\to 0} F_P,$$

where F_P is the component of the force at P resolved along the tangent to the path by which P approaches O.

To prove this we must show that if η be any arbitrary small quantity we can always choose a point K on the curve by which P approaches O such that for all positions of P on the curve between K and O

$$\left| \frac{V_O - V_P}{OP} - F_O \right| < \eta,$$

where F_O represents $\lim\limits_{OP\to 0} F_P$.

Let us regard η as the sum of three arbitrary parts η_1, η_2, and η_3.

We take a point J on the curve between P and O, and notice that

$$\frac{V_O - V_P}{OP} - F_O = \frac{V_O - V_J}{OP} + \frac{V_J - V_P}{JP} \cdot \frac{JP}{OP} - F_O;$$

and, remembering that

$$V_J - V_P = \int_P^J F ds,$$

where F is the tangential force and ds an element of the curve, we apply the first theorem of mean value and so get

$$V_J - V_P = JP \, . \, F_Q,$$

where Q is some point on the curve between J and P.

Thus

$$\frac{V_O - V_P}{OP} - F_O = \frac{V_O - V_J}{OP} + \frac{JP}{OP} \cdot F_Q - F_O.$$

Since F is definite at all points between O and P, and has a definite limit for a point tending to coincidence with O, there is a definite superior limit to the absolute value of F for the points of the curve lying between O and any definite point K; we call this superior limit M.

Now we choose K so near to O that, for all points P between K and O, $|F_P - F_O| < \eta_1$ and therefore also $|F_Q - F_O| < \eta_1$; this we can do because F_P has the limit F_O.

We next take P anywhere on OK, and P having been chosen, we can choose a point L_2 so that for all points J between L_2 and O, $|V_O - V_J| < OP \, . \, \eta_2$, this being possible because V_J has the limit V_O. And a point L_3 can be chosen so that for all points J between L_3 and O

$$\frac{JP}{OP} = 1 + \epsilon$$

where $|\epsilon| < \eta_3/M$. We now take J to be between O and the nearer of the points L_2, L_3.

Thus

$$\frac{V_O - V_J}{OP} + \frac{JP}{OP} F_Q - F_O = \frac{V_O - V_J}{OP} + (1 + \epsilon) F_Q - F_O$$

$$= \left[\frac{V_O - V_J}{OP}\right] + [F_Q - F_O] + \left[\epsilon M \, . \, \frac{F_Q}{M}\right],$$

where we notice that $|F_Q| < M$. The modulus of the first expression in square brackets is less than η_2, that of the second is less than η_1, that of the third is less than η_3, hence the modulus of the sum of the three expressions is less than $\eta_1 + \eta_2 + \eta_3$ or η. Thus we have been able to choose K so that for all points P on the curve between O and K

$$\left| \frac{V_O - V_P}{OP} - F_O \right| < \eta,$$

which establishes the theorem.

IX. Volume Integrals through regions that extend to infinity.

36. The integrals to which we have so far been devoting most attention are those whose peculiarity consists in the subject of integration becoming infinite at a point in the range. Another kind of integral requiring special study occurs frequently in mathematical physics, namely, a volume integral taken through a region which extends to infinity.

By the integral $\int f\,d\tau$ taken through all space outside certain finite closed surfaces S_1, S_2, etc. is meant the limit of the integral taken through a region bounded internally by S_1, S_2, etc., and externally by a surface B, as the linear dimensions of B and the distances of all its points from the inner boundaries become indefinitely great, provided such a limit exists and is independent of the shape of B. When the limit exists and is independent of the shape of B the integral is said to be convergent; if the limit has a finite value which is not independent of the shape of B, the integral is said to be semi-convergent.

The following is the chief test of convergence. *If we measure r from some fixed origin, and if f is such that, for all values of r greater than a definite length a, f is less in absolute value than $Mr^{-\mu}$ where M is a constant and $\mu > 3$, the integral is convergent.* We shall prove this by shewing that, corresponding to any arbitrarily chosen small quantity σ, there can always be found a closed surface θ surrounding O and all the surfaces S_1, S_2, etc., such that all closed surfaces t surrounding θ have the property that

$$\left| \int_\theta^t f\,d\tau \right| < \sigma.$$

Take for the surface θ a sphere $r = \eta$ large enough to surround the sphere $r = a$ and all the inner boundaries of the region; and let ω be

the distance from O to the furthest point of the outer boundary t. Then

$$\left|\int_\theta^t f\,d\tau\right| \leqslant \int_\theta^t |f|\,d\tau, \quad \leqslant \int_\theta^\omega |f|\,d\tau,$$

the upper limit in the last integral being the sphere $r=\omega$. Thus

$$\left|\int_\theta^t f\,d\tau\right| < M\int_\theta^\omega r^{-\mu}\,d\tau, \quad < 4\pi M\int_\eta^\omega r^{2-\mu}\,dr,$$

$$< \frac{4\pi M}{\mu-3}\left(\eta^{-(\mu-3)} - \omega^{-(\mu-3)}\right), \text{ a positive quantity,}$$

$$< \frac{4\pi M}{\mu-3}\eta^{-(\mu-3)}.$$

Hence by choosing η greater than $\{4\pi M/(\mu-3)\,\sigma\}^{\frac{1}{\mu-3}}$ we get a surface θ satisfying the specified condition; the integral is accordingly convergent. It will be noticed that there is no restriction on the shape of the outer boundary t.

Generally speaking, if f is zero at infinity of an order higher than r^{-3} the integral is convergent; if the zero is just of the order r^{-3}, the integral may be semi-convergent or divergent.

37. When Green's theorem and allied theorems are applied to volume integrals of this type, the outer boundary which tends to become infinitely large must not be left out of account, and so we have limits of surface integrals which are spoken of as integrals over the surface infinity. If the subject ψ of integration is, for values of r greater than a finite length a, less in absolute value than $Mr^{-\mu}$, where M is finite and $\mu>2$, then $|\int\psi dS|$ taken over the sphere $r=\omega$ is less than $M\omega^{2-\mu}\int\int\sin\theta\,d\theta\,d\phi$, which has the limit zero for $\omega\to\infty$. Thus the surface integral vanishes if ψ is zero at infinity of a higher order than r^{-2}. If ψ is zero of the order r^{-2} the limit of the surface integral may be different for different shapes of B; if this is the case there is of course corresponding semi-convergence of one of the volume integrals, and special investigation is required.

38. The differentiation with respect to a parameter of a volume integral through a region extending to infinity, involving as it does two distinct passages to limits, requires special consideration. Let us consider the case in which the parameter ξ affects the subject of integration, but does not affect the specification of the inner boundaries S_1, S_2, etc. Let us suppose that $\partial f/\partial\xi$ (or f') exists and is uniformly continuous through all finite portions of the region of integration for all values of ξ considered, and that the integral of f is convergent;

and further that, for all values of ξ considered and for all values of r greater than a, there is an inequality $|f'| < Mr^{-\mu}$, where $\mu > 3$, and M, μ, and a are constants whose values do not depend on the value of ξ*. Take the outer boundary to be the sphere $r = \omega$; then

$$\frac{\partial}{\partial \xi}\int f\, d\tau - \int f'\, d\tau$$

$$= \operatorname*{Lim}_{\triangle\xi \to 0}\ \operatorname*{Lim}_{\omega \to \infty}\left[\frac{1}{\triangle\xi}\int^{\omega}\{f(\xi + \triangle\xi) - f(\xi)\}\, d\tau - \int^{\omega} f'(\xi)\, d\tau\right] \quad \ldots(28),$$

and this, by the theorem of mean value,

$$= \operatorname*{Lim}_{\triangle\xi \to 0}\ \operatorname*{Lim}_{\omega \to \infty}\int^{\omega} \epsilon\, d\tau,$$

where $\epsilon \equiv f'(\xi + \theta \triangle \xi) - f'(\xi)$ and $1 > \theta > 0$, and the notation implies that first $\omega \to \infty$ and afterwards $\triangle\xi \to 0$. If we can shew that the subject of this double limit can, by first making $\omega \to \infty$, and afterwards taking $\triangle\xi$ sufficiently small, be made less than any assigned small quantity σ, clearly the double limit will be zero.

Now since f' satisfies the conditions of the theorem of § 36, and moreover in such a way that M, μ, and a are independent of ξ, it is clear by the reasoning of that Article that, for all values of ξ considered and therefore in particular for all possible values of $\xi + \theta \triangle \xi$, we can choose a definite length η such that for all values of ω greater than η

$$\left|\int_{\eta}^{\omega} f'(\xi + \theta \triangle \xi)\, d\tau\right| \text{ and } \left|\int_{\eta}^{\omega} f'(\xi)\, d\tau\right|$$

are both less than any assigned small quantity, which we shall take to be $\frac{1}{3}\sigma$. Hence

$$\operatorname*{Lim}_{\omega \to \infty}\left|\int_{\eta}^{\omega} f'(\xi + \theta \triangle \xi)\, d\tau\right| < \tfrac{1}{3}\sigma$$

and

$$\operatorname*{Lim}_{\omega \to \infty}\left|\int_{\eta}^{\omega} f'(\xi)\, d\tau\right| < \tfrac{1}{3}\sigma.$$

And η being chosen and therefore finite, however large, the uniform continuity of f' ensures our being able to choose a value of $\triangle\xi$ such that for it and for all smaller values $|\epsilon|$ is less than an arbitrary small quantity; this small quantity we choose to be $\frac{1}{3}\sigma T^{-1}$, where T is the finite volume $\int^{\eta} d\tau$. This makes $\left|\int^{\eta} \epsilon\, d\tau\right| < \frac{1}{3}\sigma$. Thus

$$\operatorname*{Lim}_{\omega \to \infty}\left|\int^{\omega} \epsilon\, d\tau\right| = \operatorname*{Lim}_{\omega \to \infty}\left|\int_{\eta}^{\omega} f'(\xi + \theta \triangle \xi)\, d\tau - \int_{\eta}^{\omega} f'(\xi)\, d\tau + \int^{\eta} \epsilon\, d\tau\right| < \sigma.$$

* We can get greater generality by simply requiring that the integral of f' shall be *uniformly* convergent for the contemplated range of values of ξ; but it seems better not to introduce into the text the idea of uniform convergence, especially as there are additional difficulties in the proof.

Hence the double limit on the right-hand side of (28) is zero, and therefore the differentiation of the integral of f is effected by the rule of differentiating under the sign of integration.

Differentiation with respect to a parameter which affects only the specification of one of the inner boundaries, say S_1, clearly gives rise merely to a surface integral over S_1.

39. Volume integrals through regions extending to infinity occur in electrical theory as expressions for electrostatic and for electrodynamic energy, and in other ways. They occur in the theory of gravitation and electrostatic potential in proofs of the important "theorems of uniqueness." They occur in Hydrodynamics as representing kinetic energy, and "impulse." Differentiation of such integrals is employed in the dynamical theory of solid bodies moving through an infinitely extended liquid. In every such application of these integrals it is necessary to make sure that there is such convergence as will render the formulae valid.

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